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Courant Institute of
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Magneto-Fluid Dynamics Division

Problems in Magnetostatic Equilibrium

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Introduction

Magneto-fluid dynamics and plasma physics are a continuing source of nonstandard and unusual mathematical structures (e.g. see Refs.[1],[2],[3]). Once such question arises in a hyperbolic system in which the time-like direction is periodic or re-entrant. The problem which we consider here is that of a composite system, partly elliptic and partly degenerate hyperbolic (i.e. with real characteristics that are lines instead of cones), in a toroidal domain. It has been found in this case that it is impossible to formulate a general boundary value problem with smooth solutions [4]. There is an open question of the possibility of a formulation in terms of appropriate weak solutions; also an important practical question is the interpretation of the large literature of formal calculations which give explicit answers where none exist!

The Differential Equations

The problem is the static balance of a perfectly conducting plasma in equilibrium with a magnetic field. In a macroscopic, scalar pressure, MHD version we have

$$\nabla p = J \times B \tag{1}$$

$$J = \text{curl } B, \quad \text{div } B = 0$$

The characteristics of this system are one elliptic cone (as in the special case, $\text{curl } B = 0$) plus the magnetic lines counted twice as degenerate real characteristics (c.f. streamlines in fluid flow).

It is frequently convenient to parameterize $\text{div } B = 0$ locally in terms of two stream functions,

$$B = \nabla \alpha \times \nabla \beta \quad (2)$$

One of the two sets of real characteristics can sometimes be eliminated by considering $p = p(\alpha, \beta)$ to be a known function; i.e. by assigning a value of p to each magnetic line--but this may be troublesome in the large.

In an anisotropic, guiding center version, instead of $p = p(\alpha, \beta)$ we introduce $p = p(\alpha, \beta, B)$ where the absolute value of the field strength, B , is taken as the third coordinate instead of arc length along a magnetic line. We obtain the system [3]

$$\begin{aligned} \frac{\partial p}{\partial \alpha} \nabla \alpha + \frac{\partial p}{\partial \beta} \nabla \beta &= K \times B \\ K &= \text{curl } (\sigma B), \quad \text{div } B = 0 \\ \sigma &= 1 - \frac{1}{B} \frac{\partial p}{\partial B} \end{aligned} \quad (3)$$

If $p(\alpha, \beta, B)$ is considered to be given, the family of magnetic field lines remains as one real characteristic, and there is an additional nondegenerate cone which is elliptic provided that $\sigma > 0$ and $\partial(\sigma B)/\partial B > 0$,

$$\begin{aligned} \frac{1}{B} \frac{\partial p}{\partial B} &< 1 \\ \frac{\partial}{\partial B} \left(\frac{1}{B} \frac{\partial p}{\partial B} \right) &< 1 \end{aligned} \quad (4)$$

If these inequalities (which refer to the 'firehose' and 'mirror' instabilities respectively) are violated, (3) does not give a well-posed boundary value problem for B [5].

In terms of conventional physical variables, the quantity we call p is $p_{||}$, the parallel component of the anisotropic pressure tensor; $p_{\perp} = -B^2 \partial(p/B)/\partial B$ is the perpendicular component, and $\sigma - 1 = (p_{\perp} - p_{||})/B^2$ is a measure of the anisotropy. The physical condition that $p_{\perp}$ be positive, $\partial(p/B)/\partial B < 0$, implies the firehose condition when $p < B^2$ and is implied by it otherwise.

With axial symmetry, writing

$$B = \frac{n}{r} \times \nabla \psi + n B_{\theta} \quad (5)$$

we find that both real characteristics integrate out explicitly, one through $p = p(\psi, B)$ and the other through

$$\frac{1}{2} \sigma r^2 B_{\theta}^2 = g(\psi) \quad (6)$$

to give a nominally elliptic equation

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sigma} \left(\frac{\partial \psi}{\partial z} \frac{\partial \sigma}{\partial z} + \frac{\partial \psi}{\partial r} \frac{\partial \sigma}{\partial r} \right) \\ + \frac{1}{\sigma^2} g'(\psi) + \frac{r^2}{\sigma} \frac{\partial p}{\partial \psi} = 0 \end{aligned} \quad (7)$$

[the given function, $\sigma = \sigma(\psi, B)$, where $B^2 = |\nabla \psi|^2/r^2 + 2g/\sigma r^2$, determines σ implicitly in terms of r, ψ , and $|\nabla \psi|$].

For a more detailed discussion of the guiding-center equations, see Refs. [3] and [5].

Boundary Value Problems

A plausible theory can be given, based in part on proved existence theorems, for a tubular domain with end surfaces on which $B_n > 0$ and $B_n < 0$ respectively and a lateral surface where $B_n = 0$. The domain is assumed to be simply covered by the direction field B . Corresponding to the elliptic characteristic cone we specify the normal component, B_n , subject to $\oint B_n dS = 0$, on the entire boundary. Corresponding to the real characteristics, we specify two pieces of initial data at one end of the tube (or one at each end); e.g. we may specify p and J_n . In addition to qualitative arguments such as counting characteristics, we confirm a consistent variational description with imposed or natural boundary data, we verify actual existence in the case of axial symmetry, and we are able to construct plausible iteration procedures which rationally account for all boundary data and which can be shown to converge with suitable restrictions in axial symmetry (and should probably converge more generally); see Ref.[6].

Similar conjectures have been made for a toroidal domain [6] (and without even recognizing them as conjectures [7]), assuming that the domain is covered by a family of nested toroidal pressure surfaces. In the absence of an 'initial' surface which intersects each magnetic line, two pieces of information, e.g. $p(\psi)$ and the rotation number $\alpha(\psi)$, are assigned to each flux surface. The plausibility arguments such as characteristic counting, variational analysis, and actual proved examples in axial symmetry are equally valid for the torus. In addition,

the fact that the tangent field B satisfies an elliptic equation on each pressure surface refines the counting by demanding that two periods be assigned to each pressure surface.

A small shadow was cast even at the beginning, by the fact that iterations that were plausible in the open tube could not be made to work for the torus. After a series of disturbing irregularities in expansions (e.g. see [8]), it was finally recognized that axial (or other) symmetry is not just a convenience for calculation, but it is necessary for existence [4].

The situation is qualitatively similar to the question of existence of flux surfaces for harmonic vector fields, $\text{curl } B = 0$, in general toroidal domains, but it is not the same. The possibility of taking $J \neq 0$ allows more flexibility. For example, given an arbitrary set of nested flux surfaces, the necessary condition, $\nabla p \cdot B \times \text{curl } B = 0$, $\text{div } B = 0$, can be satisfied together with arbitrarily given $p(\psi)$ and $\alpha(\psi)$. The problem of finding equilibria is, also connected with (but again is not a simple consequence of) the pathology of particle orbits in a toroidal domain. It is most directly related to the very singular guiding center limit which permits infinite currents to develop [4].

Two problems immediately present themselves. The first is how to alter the mathematical formulation in order to restore existence. The second is how to interpret the large literature of formal calculations in specific geometries which give explicit results--but even in cases where no correct answer exists.

With regard to restoring existence, the microscopic model which leads to the anisotropic system (3) is helpful in associating mathematical nonexistence with actual physical pathology [4] (in some cases, this has been observed experimentally). In other words, there are cases in which we cannot expect help from any improved physical model; nonexistence of a stationary solution is the correct physical prediction. Qualitatively, we are able to draw the conclusion that in the limit of small Larmor radius, a smooth pressure profile must be associated with pathological current (J is nonintegrable). On the other hand, assuming that J remains bounded implies that ∇p is pathological--at best measurable, but never continuous. In other words, p and B cannot both be smooth. We are presented with an unusual situation in partial differential equations where weak solutions are not adopted as a mathematical convenience but at the insistence of the equations.

It is interesting to note that the theory of Arnold [9] in this problem, although technically correct, is vacuous because it assumes analyticity of p and of B .

The question of the existence of weak solutions to the systems (1) or (3) is open (although some preliminary analysis has been done in weak perturbations of a given strong solution [10]). To find a weak solution, it is first necessary to specify data such as $p(\psi)$ in situations where the variable ψ is not well-defined. We may consider, in accordance with the theories of Arnold [11] and Moser [12], that there is a family of surfaces, $\psi = \text{const.}$, of positive

measure, together with gaps (ergodic, islands, etc.) surrounding regions of rational rotation number, α . We can require that p be constant in each gap. It is also plausible that α is constant throughout each gap (it is, of course, constant in any island). A smooth function $p(\alpha)$ might therefore be specified as a constraint which could be well-defined even for weak solutions. Such an analysis has not yet been done.

The second problem mentioned above is the interpretation of formal calculations, usually expansions, which present ostensibly valid equilibria, complete with flux surfaces. In some cases the pathology is hidden in a higher order of expansion (e.g. if a gap has thickness ϵ^n). In other cases, the pathology appears to be transcendental and is not seen to any order.

The salvation of such calculations rests on the fact that the time-dependent MHD initial value problem is well-posed. Pathology enters only when we ask the improper question of the existence of a static (or even stationary) solution.

The existence of static solutions is a very great convenience in classical analysis. For example, almost all stability analysis introduces a small perturbation from a postulated equilibrium. We can justify such MHD stability calculations as follows.

The equations of motion are hyperbolic with characteristics determined by the instantaneous state, viz. $p(x)$, $u(x)$, and $B(x)$. Suppose we are given truncated power series,

$$p^{(n)} = \sum_0^n \epsilon^r p_r^{(n)}, \quad B^{(n)} = \sum_0^n \epsilon^r B_r^{(n)}, \quad \text{such that}$$

$$\nabla p^{(n)} = J^{(n)} \times B^{(n)} + \epsilon^{n+1} R_{n+1} \quad (8)$$

and $R_{n+1}(x, \epsilon)$ is a known smooth function of x . The problem is to confirm whether the solution of the hyperbolic system with initial values $u = 0$, $p = p^{(n)}$, $B = B^{(n)}$ remains near the initial state for an appreciable time.

As a simplification, consider the symmetric hyperbolic system obtained by a conventional linearization of the equation of motion about the state $u = 0$, $p = p^{(n)}$, $B = B^{(n)}$. This system is inhomogeneous through R_{n+1} , has zero initial values, and has fixed characteristics determined by $p^{(n)}$ and $B^{(n)}$. The solutions will remain bounded (in a suitable norm) if the system is stable in a conventional sense. The crucial point is that the theory of the hyperbolic system is standard when linearized about $p^{(n)}$ and $B^{(n)}$ whether or not this state corresponds to a true static equilibrium.

At present, the information available is more limited. The stability analysis is itself carried out as an expansion in ϵ . Since in the most common case there is a highly degenerate eigenvalue, $\omega = 0$, which is spread by the parameter ϵ , even the linear program mentioned above has not been carried out. But it is at least possible to point to a conventional mathematical framework within which we could determine the significance of formal expansions of nonexistent equilibria.

There is one sense in which the formal expansions do reflect a shadow of the nonexistence of a legitimate solution. In common with many asymptotic expansions, the determination of a given order of expansion is not complete until partial information is extracted from a higher order. In the present case, we find that field perturbations of order ϵ^{n+2} and sometimes even ϵ^{n+4} react back on the equilibrium configuration to order ϵ^n . The (nonexistent) calculated configuration is therefore very sensitive to perturbations [13].

The Free Boundary

There is one special case in which the nature of the weak equilibrium can be ascertained. The MHD free boundary is a flux surface separating a harmonic field, $\text{curl } B = 0$, $\text{div } B = 0$, from a field-free region at constant pressure p , and is mathematically identical to the classical, fluid free boundary in virtue of the boundary condition, $p = (1/2)B^2$ or $B^2 = \text{const.}$

There is a large literature concerning two-dimensional and axially symmetric free boundaries. One attack which we can hope to extend to three dimensions is the inverse method. Given an analytic curve on which $\psi = 0$ and $\partial\psi/\partial n = 1$, we employ the Cauchy-Kovalevsky theorem to solve an initial value problem for $\Delta\psi = 0$ in the neighborhood of the curve.

In three dimensions, there is no equation equivalent to $\Delta\psi = 0$. To use the potential, $B = \nabla\phi$, $\Delta\phi = 0$, we must

first find, on a given analytic surface, a scalar ϕ with constant surface gradient, $|\nabla\phi| = 1$. The Cauchy initial values will then be ϕ and $\partial\phi/\partial n = 0$.

This is an essentially hyperbolic problem on the initial surface. The Hamilton-Jacobi wave fronts, $\phi = \text{const.}$, are orthogonal to geodesic rays (magnetic field lines on the surface). In other words, if on a given analytic toroidal surface, we are given an analytic single-valued field of geodesics, we can then construct the orthogonal trajectories and assign a multivalued function ϕ with single-valued $\nabla\phi$ and $|\nabla\phi| = 1$. Existence of a three-dimensional field $B = \nabla\phi$ in the neighborhood of the surface then follows.

The question reduces to that of finding a geodesic field--this, of course, is not always possible. If we treat the problem by perturbation about an elementary surface (e.g. a cylinder with identified ends or an axially symmetric surface) on which we know of the existence of a geodesic field for each value of the rotation number α , the problem is reduced to an application of the twist mapping theorem of Moser and Arnold. In other words, there will be a geodesic field on any surface which is close to an 'elementary' surface for a set of α of positive measure which avoids the neighborhood of rational α .

From this result, we can conjecture the structure of the solution of the direct problem. In two dimensions,

for a given fixed outer boundary, there exists a one parameter family of interior free boundaries, e.g. parametrized by area. In three dimensions, let us choose α as the parameter instead of area (or volume). We can expect to find a family of free boundaries corresponding to a set of positive measure of α , but not a complete family, for each value of the area, as in two dimensions.

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